

SPECTRA OF LINEAR FRACTIONAL COMPOSITION OPERATORS ON $H^2(B_N)$

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ABSTRACT. We characterize the spectra of composition operators on the Hardy space $H^2(B_N)$, when the symbols are elliptic or hyperbolic linear fractional self-maps of B_N . Therefore, combining with the result obtained by Bayart [4], the spectra of all linear fractional composition operators on $H^2(B_N)$ are completely determined.

1 Introduction

Let B_N denote the unit ball of \mathbb{C}^N , and let $\varphi : B_N \rightarrow B_N$ be an analytic map. In this paper, we consider the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$, acting on the classical Hardy space $H^2(B_N)$. For $N > 1$, some authors gave the examples of unbounded composition operators on $H^2(B_N)$ (see Section 3.5 of [11]), which exhibit surprisingly different behaviors with the case of one complex variable. So many properties of composition operators in several variables are not easily managed.

However, if the symbol φ is a linear fractional self-map of B_N , that is,

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + d}$$

for an $N \times N$ matrix $A = (a_{jk})$, two column vectors $B = (b_j)$, $C = (c_j)$ of $N \times 1$ and $d \in \mathbb{C}$, such that φ maps the unit ball B_N into itself, applying Wogen's criterion [29], Cowen and MacCluer [12] proved that C_φ is bounded on $H^2(B_N)$. In recent years, linear fractional maps of the ball and their composition operators on some analytic function spaces have been developed from various aspects, for example, geometric properties [5] and classification of semigroups [6] for linear fractional maps; cyclic behavior (see [2], [4], [20]) and essential normality (see [21], [27], [33]) of their composition operators. Similar to the case of the unit disk D , linear fractional self-maps of B_N give rich examples to exhibit the complexity of properties of the associated composition operators. Moreover, motivated by the fact that the linear

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fractional model in D provides a useful tool to deal with composition operators induced by general maps, some authors began to study a linear fractional model for general maps of the unit ball, see [3] and [9].

In this paper, we are interested in the spectra of linear fractional composition operators on $H^2(B_N)$, this is a continuation of [4]. As a part of operator theory, the structures of spectra of composition operators are very important. Several authors have used this tool to investigate the cyclicity of composition operators (see [4], [14]). In the unit disk, the spectra of invertible and compact composition operators on $H^2(D)$ have been described completely, while understanding the spectra of general composition operators is not easy. When φ is a linear fractional self-map of D , the spectrum of C_φ on $H^2(D)$ (see [8]) is well known. Recently, Higdon [15] completely gave the spectrum of C_φ on the Dirichlet space \mathcal{D} . Moreover, Hurst [16] obtained some results for the spectrum of C_φ on some weighted Hardy spaces. These results not only show the diversity of spectra of composition operators, and much of the spectral information depends on the behavior of φ near the Denjoy-Wolff point. From the table of [8], one has the following results.

Theorem A. *Suppose that φ is a linear fractional self-map of D , not automorphism. The spectrum of C_φ on $H^2(D)$ can be described as follows:*

(i) *If φ has an interior fixed point a , there exists two cases. When φ fixes a boundary point τ , then*

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \varphi'(\tau)^{-1/2}\} \cup \{1\}.$$

Otherwise, C_φ^n is compact for some positive integer n and

$$\sigma(C_\varphi) = \{\varphi'(a)^k : k = 0, 1, \dots\} \cup \{0\}.$$

(ii) *If φ is parabolic, by the Cayley transformation, φ is conjugated to a map $\psi(z) = z + t$ on the upper-half plane, then*

$$\sigma(C_\varphi) = \{e^{\beta t} : \beta \leq 0\} \cup \{0\}.$$

(iii) *If φ is hyperbolic with the Denjoy-Wolff point $\tau \in \partial D$, then*

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \varphi'(\tau)^{-1/2}\}.$$

In the unit ball, only the spectra of automorphism-induced composition operators and compact operators have been determined (see Chapter 7 in [11]). For other cases, less results have been obtained, even for the symbols being linear fractional self-maps of B_N . Recently, Jury [22] has computed the spectral radii of linear fractional composition operators on $H^2(B_N)$, soon later, the spectra of composition operators were characterized by Bayart [4] when their symbols are parabolic linear fractional maps of B_N . In order to illustrate these results, we recall some definitions

for the Denjoy-Wolff points and the classification of linear fractional maps of B_N . In this paper, we denote by $LFM(B_N)$ the set of linear fractional maps of B_N .

The Denjoy-Wolff Theorem of the unit ball is the following (see [25]).

Theorem B. *Let $\varphi \in LFM(B_N)$ with no fixed points in B_N . There exists a unique point $\tau \in \partial B_N$ such that $\varphi(\tau) = \tau$ and $\langle d\varphi_\tau(\tau), \tau \rangle = \alpha$ with $0 < \alpha \leq 1$.*

The point τ is called the Denjoy-Wolff point of φ and α is the boundary dilation coefficient. In particular, we call such a map hyperbolic when $\alpha \in (0, 1)$ and parabolic when $\alpha = 1$. Otherwise, if φ has a fixed point in B_N , then we call it elliptic.

Now, we have the following result for spectral radii of linear fractional composition operators on $H^2(B_N)$ (see [22]).

Theorem C. *Let $\varphi \in LFM(B_N)$. The spectral radius of C_φ acting on $H^2(B_N)$ is 1 if φ is elliptic; if φ is non-elliptic with the boundary dilation coefficient α , then the spectral radius is $\alpha^{-N/2}$.*

For a parabolic linear fractional self-map φ of B_N , considering its conjugating map ψ on the Siegel half-plane H_N , Bayart [4] obtained a normal form of ψ and gave a classification for this normal form. Based on this classification, he completely computed the spectrum of C_φ on $H^2(B_N)$.

In this paper, we continue to investigate the spectra of linear fractional composition operators on $H^2(B_N)$ induced by the remaining maps, and we organize it as follows. In Section 2, we first look for a classification for elliptic linear fractional self-maps of B_N , the idea comes from the classification of elliptic semigroups (see [6]) and the geometric classification according to their fixed point sets (see [5]). Different to the case of the disk, elliptic linear fractional maps of B_N contain three cases. Therefore, we will prove that the spectra of the associated composition operators have three different structures. For a general map φ , univalent, not automorphism, which fixes a point in D or B_N , the spectrum of C_φ on many function spaces has been studied (see [10], [24], [26], [31], [32]). It is easy to see that the proofs of these results follow the same pattern. We shall use an analogue approach to deal with the spectrum of C_φ on $H^2(B_N)$, when the symbol φ is an elliptic linear fractional map with only one boundary fixed point.

In Section 3, we devote to hyperbolic linear fractional self-maps φ of B_N . It is well known that φ has only one or two fixed points on the boundary ∂B_N (see [5]). First, we will obtain a simple form for each conjugation map ψ of φ on the Siegel half plane H_N . Applying these forms, we can completely determine the spectra of hyperbolic composition operators on $H^2(B_N)$. In particular, when φ fixes only one boundary point, the spectrum of C_φ has a similar structure with the corresponding case in the disk, which has been proved in the first author's doctoral thesis [19]. For the case φ fixing two boundary points, Xu and Deng [30] recently used the method of Bayart [4] to obtain the spectrum of C_φ on $H^2(B_N)$. Thus, the spectra of all linear fractional composition operators on $H^2(B_N)$ are completely characterized.

2 Spectra of elliptic composition operators

In this section, we deal with elliptic linear fractional self-maps of B_N and the spectra of the associated composition operators. First, we need a classification for these maps.

Recall that a slice S is a non-empty subset of B_N of the form $S = B_N \cap V$, where V is a one-dimensional affine subspace of \mathbb{C}^N . A p -dimensional slice of B_N is the non-empty intersection between B_N and a p -dimensional affine subspace of \mathbb{C}^N with $p \geq 0$. By Hervé's theorem (see [1] or [28]), if $\varphi \in LFM(B_N)$ has a non-empty fixed points set in B_N then such a set is a p -dimensional slice of B_N .

In [5], Bisi and Bracci gave a geometric classification for linear fractional self-maps of B_N based on their fixed point sets. In this classification, elliptic linear fractional maps contain three cases. For elliptic semigroup of linear fractional self-maps of B_N , Bracci et al [6] also classified this semigroup into three different cases. Surprisingly, we find that there is a close connection between two classifications and obtain the following result. Let

$$L_U(\varphi, z_0) = \bigoplus_{|\alpha|=1} \ker(d\varphi_{z_0} - \alpha I)^N$$

for $\varphi \in \text{Hol}(B_N, B_N)$ with a fixed point $z_0 \in B_N$, which is called the unitary space of φ at the point z_0 , the dimension of it is called the unitary index of φ (see [6]).

Theorem 2.1. *Let $\varphi \in LFM(B_N)$ be elliptic with a fixed point $z_0 \in B_N$ and let $p = \dim L_U(\varphi, z_0)$.*

(1) *If $p > 0$, then φ is conjugated to a map ψ with*

$$\psi(z', z'') = (Uz', Az''), \quad (z', z'') \in \mathbb{C}^p \times \mathbb{C}^{N-p} \cap B_N,$$

where U is a unitary diagonal matrix of $\mathbb{C}^{p \times p}$ and A is a matrix of order $N - p$ with $\|A\| < 1$.

(2) *If $p = 0$, φ will fix at most one boundary point. In this case:*

(i) *When φ has no boundary fixed point, then φ is conjugated to a map of the form*

$$\psi(w) = Aw$$

defined on a complex ellipsoid

$$\Delta_1 = \{w = (w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1} : \frac{1}{r^2}|w_1 - \sqrt{r^2 - 1}|^2 + |w'|^2 < r^2\}$$

for some $r \geq 1$, where A is a matrix of $\mathbb{C}^{N \times N}$ with $\|A\| < 1$.

(ii) *When φ has only one boundary fixed point, then φ is conjugated to*

$$\psi(w) = Aw$$

on a half-plane

$$\Delta_2 = \{w = (w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1} : \text{Re } 2w_1 > |w'|^2 - 1\},$$

where A is a matrix of order N with $\|A\| < 1$.

Proof. We will use a similar argument as that of Theorem 3.2 and Corollary 3.3 in [6] to obtain these results. Especially, the proof of (1) can be seen in [6] or Proposition 3.8 of [19]. Here, we only give a proof for (2).

If $p = 0$, it is clear that φ fixes only one interior point z_0 . Up to conjugation with automorphisms of B_N , we may assume that $z_0 = 0$. We claim that φ fixes at most one boundary point. Otherwise, if φ fixes two points on ∂B_N , by Theorem 3.2 of [5], φ is conjugated to a map which has a hyperbolic automorphism (not identity) at first coordinate. Thus, φ has no fixed point in B_N , which contradicts with $\varphi(0) = 0$. When φ has more than two boundary fixed points, similar to the argument in the proof of Theorem 3.1 in [5], φ will fix a q -dimensional slice of B_N with $q \geq 1$. This means that the unitary index p of φ must be larger than q , contradicting to the hypothesis $p = 0$. So that the claim is true.

Now, $\varphi(0) = 0$ gives

$$\varphi(z) = \frac{Az}{\langle z, C \rangle + 1}$$

for some $A \in \mathbb{C}^{N \times N}$ and $C \in \mathbb{C}^N$. Since φ maps B_N into B_N , we have $|C| < 1$. A computation shows that $d\varphi_0 = A$. Thus $p = 0$ implies $\|A\| < 1$, that is, $A^* - I$ is invertible. Immediately, there exists a vector $V \in \mathbb{C}^N$ such that $(A^* - I)V = C$ and $\delta := |V| \leq 1$. Conjugating φ by a unitary map U with $U^*V = \delta e_1$, where $e_1 = (1, 0, \dots, 0) \in \partial B_N$, we obtain

$$\tilde{\varphi}(z) = \frac{A_1 z}{\delta \langle z, (A_1^* - I)e_1 \rangle + 1}$$

with $A_1 = U^*AU$ and $\|A_1\| < 1$. Define

$$\sigma(z) = \frac{z}{-\delta z_1 + 1}, \quad z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{N-1} \cap B_N,$$

it is an one-to-one holomorphic linear fractional map from B_N onto $\Omega := \sigma(B_N)$. Consequently, we deduce that

$$\sigma \circ \tilde{\varphi}(z) = A_1 \sigma(z), \quad z \in B_N,$$

and

$$\Omega = \{w = (w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1} : |1 + \delta w_1|^2 > |w'|^2\}.$$

If φ fixes no boundary point, then $\delta < 1$. Otherwise, $\delta = 1$ implies that

$$\tilde{\varphi}(z) = \frac{A_1 z}{\langle z, (A_1^* - I)e_1 \rangle + 1}$$

has a boundary fixed point e_1 . So φ also fixes a boundary point, which is a contradiction. If we set $r = (1 - \delta^2)^{-1/2}$, then $r \geq 1$ and

$$\Omega = \triangle_1 = \{(w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1} : \frac{1}{r^2} |w_1 - \sqrt{r^2 - 1}|^2 + |w'|^2 < r^2\}.$$

If φ fixes only one boundary point, then $\tilde{\varphi}$ also has only one fixed point $\tau \in \partial B_N$. It follows that

$$A_1 \sigma(\tau) = \sigma \circ \tilde{\varphi}(\tau) = \sigma(\tau),$$

that is, $\sigma(\tau)$ is a fixed point of the matrix A_1 . since $\|A_1\| < 1$, we see that $\sigma(\tau) = 0$ or $\sigma(\tau) = \infty$. However, σ is an one-to-one map from B_N onto Ω and $\sigma(0) = 0$. This implies $\sigma(\tau) = \infty$. Thus, we have $-\delta\tau_1 + 1 = 0$ and $\delta = \frac{1}{\tau_1} \geq 1$, where τ_1 denotes the first coordinate of τ . Which combining with $\delta \leq 1$ yields $\delta = 1$. Hence, a calculation shows that

$$\Omega = \Delta_2 = \{(w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1} : \operatorname{Re} 2w_1 > |w'|^2 - 1\},$$

which is a domain similar to the Siegel half-plane. \square

Next, we will apply this classification to compute the spectrum of C_φ on $H^2(B_N)$, when the symbol φ is a elliptic linear fractional self-map of B_N . In the first case of Theorem 2.1, φ is conjugated to a map $\psi(z', z'') = (Uz', Az'')$. Let $\Lambda_\varphi = \{\lambda_1, \dots, \lambda_N\}$ be the union of the spectrum of U and A , and write Λ_φ^α for $\lambda_1^{\alpha_1} \cdots \lambda_N^{\alpha_N}$, where α is a multi-index of \mathbb{N}^N . In this case, we have the following result for the spectrum of C_φ , the proof is inspired by that of [4] in computing the spectra of parabolic composition operators on $H^2(B_N)$.

Theorem 2.2. *Let $\varphi \in LFM(B_N)$ be elliptic, non-automorphism, and let $p = \dim L_U(\varphi, z_0)$ for its fixed point $z_0 \in B_N$. When $p > 0$, the spectrum of C_φ on $H^2(B_N)$ is*

(i) *a union of circles if U has an irrational unimodular eigenvalue:*

$$\sigma(C_\varphi) = \bigcup_{\alpha \in \mathbb{N}^N} \Lambda_\varphi^\alpha \mathbb{T} \cup \{0\};$$

(ii) $\sigma(C_\varphi) = \bigcup_{\alpha \in \mathbb{N}^N} \Lambda_\varphi^\alpha \cup \{0\}$ *if all eigenvalues of U are rational.*

Proof. By Theorem 2.1, we see that φ is conjugated to a map with the form

$$\psi(z, w) = (Uz, Aw), \quad (z, w) \in \mathbb{C}^p \times \mathbb{C}^{N-p} \cap B_N,$$

where $U = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_p})$ and $\|A\| < 1$. Up to conjugation by a unitary map, we may assume that A is upper-triangular with the diagonal entries $\lambda_1, \dots, \lambda_{N-p}$, which obviously are the eigenvalues of A . Next, we will determine the spectrum of C_ψ .

For each $i \in \{1, \dots, N-p\}$, let $w(i) = (w_{i,1}, \dots, w_{i,N-p})^\top$ be a non-zero eigenvector of A^\top associated to λ_i , i.e. $A^\top w(i) = \lambda_i w(i)$. Write $w = (w_1, \dots, w_{N-p})$, we consider the function

$$F(z, w) = z^\beta [w w(1)]^{\gamma_1} \cdots [w w(N-p)]^{\gamma_{N-p}}, \quad (z, w) \in \mathbb{C}^p \times \mathbb{C}^{N-p} \cap B_N,$$

where $\beta = (\beta_1, \dots, \beta_p) \in \mathbb{N}^p$ and $\gamma = (\gamma_1, \dots, \gamma_{N-p}) \in \mathbb{N}^{N-p}$. It is clear that $[w w(i)]^{\gamma_i}$ is a homogeneous polynomial of w with degree γ_i . So that F belongs to $H^2(B_N)$

and

$$\begin{aligned}
F \circ \psi(z, w) &= (Uz)^\beta [wA^\top w(1)]^{\gamma_1} \cdots [wA^\top w(N-p)]^{\gamma_{N-p}} \\
&= e^{i(\beta_1\theta_1 + \cdots + \beta_p\theta_p)} \lambda_1^{\gamma_1} \cdots \lambda_{N-p}^{\gamma_{N-p}} z^\beta [w w(1)]^{\gamma_1} \cdots [w w(N-p)]^{\gamma_{N-p}} \\
&= e^{i(\beta_1\theta_1 + \cdots + \beta_p\theta_p)} \lambda_1^{\gamma_1} \cdots \lambda_{N-p}^{\gamma_{N-p}} F(z, w).
\end{aligned}$$

Thus, for any multi-index $\alpha = (\beta, \gamma) \in \mathbb{N}^N$, $e^{i(\beta_1\theta_1 + \cdots + \beta_p\theta_p)} \lambda_1^{\gamma_1} \cdots \lambda_{N-p}^{\gamma_{N-p}}$ is an eigenvalue of C_ψ with the eigenvector F . Hence, $\sigma(C_\psi)$ contains the set $\bigcup_{\alpha \in \mathbb{N}^N} \Lambda_\varphi^\alpha \cup \{0\}$. If U has an irrational eigenvalue $e^{i\theta_j}$, then the set $\{e^{ik\theta_j} \lambda_1^{\gamma_1} \cdots \lambda_{N-p}^{\gamma_{N-p}} : k \in \mathbb{N}\}$ is dense in the circle $|\lambda_1|^{\gamma_1} \cdots |\lambda_{N-p}|^{\gamma_{N-p}}$. Note that this set is contained in $\sigma(C_\psi)$ and $\sigma(C_\psi)$ is closed, it follows that $\sigma(C_\psi)$ contains $\bigcup_{\alpha \in \mathbb{N}^N} \Lambda_\varphi^\alpha \mathbb{T} \cup \{0\}$.

For another direction, as in the proof of Lemma 5.4 in [4], we need the following notations. For $\gamma \in \mathbb{N}^{N-p}$, let H_γ be the set of all functions F in $H^2(B_N)$, which have the form $F_\gamma(z)w^\gamma$. It is clear that $F_\gamma(z)w^\gamma$ and $F_\beta(z)w^\beta$ are orthogonal if $\gamma \neq \beta \in \mathbb{N}^{N-p}$. So we have the orthogonal decomposition $H^2(B_N) = \bigoplus_{\gamma \in \mathbb{N}^{N-p}}^\perp H_\gamma$. If we set $K_n = \bigoplus_{|\gamma| \geq n}^\perp H_\gamma$ for any $n \geq 0$, then $H^2(B_N) = \bigoplus_{|\gamma| < n} H_\gamma \oplus K_n$. Write $F = F_\gamma(z)w^\gamma$, we see that

$$F \circ \psi(z, w) = F_\gamma(Uz) \prod_{j=1}^{N-p} \left(\lambda_j w_j + \sum_{k>j} a_{j,k} w_k \right)^{\gamma_j}.$$

After introducing a natural order on the set \mathbb{N}^{N-p} (see [11, p.26] or [4]), if we use this ordering for the decomposition $H^2(B_N) = \bigoplus_{|\gamma| < n} H_\gamma \oplus K_n$, then the matrix of C_ψ is upper-triangular. Set $\rho(z, w) = (Uz, w)$, let T_γ and S_γ respectively denote the diagonal blocks of C_ψ and C_ρ corresponding to H_γ , then we have $T_\gamma = \lambda_1^{\gamma_1} \cdots \lambda_{N-p}^{\gamma_{N-p}} S_\gamma$. Since ρ is an elliptic automorphism of B_N , by Theorem 7.6 of [11], the spectrum of C_ρ on $H^2(B_N)$ is the closure of all possible products of the eigenvalues of U , which denoted by X . If U has an irrational eigenvalue of modulus 1, then $X = \mathbb{T}$. Applying Lemma 7.17 of [11] or Lemma 5.3 of [4], we get $\sigma(S_\gamma) \subset X$ and

$$\begin{aligned}
\sigma(C_\psi) &\subset \bigcup_{|\gamma| < n} \sigma(T_\gamma) \cup \sigma(C_{\psi|K_n}) \\
&\subset \bigcup_{|\gamma| < n} \lambda_1^{\gamma_1} \cdots \lambda_{N-p}^{\gamma_{N-p}} X \cup \sigma(C_{\psi|K_n}).
\end{aligned}$$

Since the spectrum of $C_{\psi|K_n}$ must be contained in the disk of radius $\|C_{\psi|K_n}\|$, we will obtain the desired result if we can show that $\|C_{\psi|K_n}\|$ tends to zero as n tends to infinity.

Let $A = T\Sigma V$ be a singular value decomposition of A , where T, V are unitary and Σ is diagonal with the diagonal entries μ_1, \dots, μ_{N-p} . We see that μ_1, \dots, μ_{N-p} are the non-negative square roots of the eigenvalues of AA^* and $\mu = \max\{\mu_1, \dots, \mu_{N-p}\} < 1$. Set $\phi_1(z, w) = (Uz, Vw)$, $\psi_\Sigma(z, w) = (z, \Sigma w)$ and $\phi_2(z, w) = (z, Tw)$, so that $C_\psi = C_{\phi_1} C_{\psi_\Sigma} C_{\phi_2}$. Note that ϕ_1 and ϕ_2 are automorphisms of B_N , it follows that

C_{ϕ_1} and C_{ϕ_2} are invertible on $H^2(B_N)$. on the other hand, K_n and K_n^\perp are stabled by C_{ϕ_1} and C_{ϕ_2} . Therefore, we only need to calculate that $\|C_{\psi_\Sigma}|_{K_n}\|$ goes to zero as $n \rightarrow \infty$.

Let $F = \sum_{|\gamma| \geq n} F_\gamma(z) w^\gamma \in K_n$. Composing it by ψ_Σ , we get

$$F \circ \psi_\Sigma(z, w) = \sum_{|\gamma| \geq n} F_\gamma(z) (\Sigma w)^\gamma = \sum_{|\gamma| \geq n} \mu_1^{\gamma_1} \cdots \mu_{N-p}^{\gamma_{N-p}} F_\gamma(z) w^\gamma.$$

If we denote $\zeta = (\zeta', \zeta'') \in \mathbb{C}^p \times \mathbb{C}^{N-p} \cap \partial B_N$, then

$$\begin{aligned} \|C_{\psi_\Sigma} F\|^2 &= \sup_{0 < r < 1} \int_{\partial B_N} |F \circ \psi_\Sigma(r\zeta)|^2 d\sigma(\zeta) \\ &= \sup_{0 < r < 1} \sum_{|\gamma| \geq n} \int_{\partial B_N} |r^{|\gamma|} \mu_1^{\gamma_1} \cdots \mu_{N-p}^{\gamma_{N-p}} F_\gamma(r\zeta') (\zeta'')^\gamma|^2 d\sigma(\zeta) \\ &\leq \sum_{|\gamma| \geq n} \mu^{|\gamma|} \sup_{0 < r < 1} \int_{\partial B_N} |F_\gamma(r\zeta') (r\zeta'')^\gamma|^2 d\sigma(\zeta) \leq \mu^n \|F\|^2. \end{aligned}$$

Here and other places in this paper, we let $\|\cdot\|$ denote the norm of $H^2(B_N)$. Applying $\mu < 1$ to the above inequality and let n go to infinity, we get the desired conclusion.

□

In the second case, we find that C_φ^n is a compact operator for some positive integer n , so that the spectrum of C_φ is easily known.

Theorem 2.3. *Let $\varphi \in LFM(B_N)$ be elliptic, non-automorphism, and let $p = \dim L_U(\varphi, z_0)$ for its fixed point $z_0 \in B_N$. If $p = 0$ and φ fixes no boundary point. Then the spectrum of C_φ on $H^2(B_N)$ is the set consisting of 0, 1 and all possible products of the eigenvalues of $\varphi'(z_0)$.*

Proof. Applying Theorem 2.1, φ is conjugated to a linear fractional map ψ of B_N with

$$\sigma \circ \psi = A\sigma,$$

where σ is an one-to-one holomorphic map from B_N onto a complex ellipsoid Δ_1 and A is a matrix of $\mathbb{C}^{N \times N}$ with $\|A\| < 1$. Moreover, in the proof of Theorem 2.1, we have $\psi(0) = 0$ and $\sigma(0) = 0$.

Let ψ^n denote the n -th iterate of ψ , then $\psi^n = \sigma^{-1} A^n \sigma$ and $\psi^n(0) = 0$. We claim that there exists a positive integer n so that C_{ψ^n} is compact. Since the set $\overline{\Delta_1} = \sigma(\overline{B_N})$ is compact in \mathbb{C}^N and $\|A\| < 1$, we see that $\sup\{|A^n \sigma(z)| : z \in \overline{B_N}\}$ goes to zero as n tends to infinity. Thus, there exist a positive integer M and a positive constant $r < 1$ such that

$$\sup_{z \in \overline{B_N}} |\sigma^{-1} A^n \sigma(z)| \leq r$$

holds for all $n \geq M$. That is $\|\psi^n\|_\infty \leq r < 1$ for such n . As we know, a linear fractional map ϕ of B_N is compact on $H^2(B_N)$ if and only if $\|\phi\|_\infty < 1$. It follows

that C_{ψ^n} is compact. Thus, by Theorem 7.2 of [11], the spectrum of C_{ψ^n} is the set consisting of 0, 1 and all possible products of the eigenvalues of $(\psi^n)'(0) = \psi'(0)^n$. On the other hand, the spectral mapping theorem gives $[\sigma(C_\psi)]^n = \sigma(C_{\psi^n}) = \sigma(C_{\psi^n})$. Hence, the spectrum of C_ψ has the desired structure. \square

For the last case in Theorem 2.1, the structure of the spectrum of C_φ on $H^2(B_N)$ is similar to that in one variable with the symbol fixing a boundary point (see Theorem A). So we first need to compute the essential spectral radius of C_φ . For $\zeta \in \partial B_N$, we will use the notation

$$d_\varphi(\zeta) = \liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}$$

for an analytic map φ of B_N into itself.

Lemma 2.4. *Let $\varphi \in LFM(B_N)$ be elliptic, non-automorphism, and let $p = \dim L_U(\varphi, z_0)$ for its fixed point $z_0 \in B_N$. If $p = 0$ and φ fixes only one boundary point. Then the essential norm of C_φ on $H^2(B_N)$ satisfies the following inequalities:*

$$\limsup_{|z| \rightarrow 1} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\frac{N}{2}} \leq \|C_\varphi\|_e \leq C \limsup_{|z| \rightarrow 1} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\frac{N}{2}}$$

for a positive constant C . So the essential spectral radius of C_φ is

$$r_e(C_\varphi) = \lim_{n \rightarrow \infty} \left(\limsup_{|z| \rightarrow 1} \left(\frac{1 - |z|^2}{1 - |\varphi^n(z)|^2} \right)^{N/2} \right)^{\frac{1}{n}}.$$

Proof. Conjugation by a unitary map, we may assume that φ fixes 0 and the boundary point e_1 . We first claim that $|\varphi(\zeta)| < 1$ for any boundary point $\zeta \neq e_1$. Otherwise, if there exist two boundary points $\zeta \neq e_1$ and η such that $\varphi(\zeta) = \eta$, we will get a contradiction.

Let $L(e_1, \zeta) = \{ce_1 + (1 - c)\zeta : c \in \mathbb{C}\} \cap B_N$ denote the one-dimensional affine subset of B_N determined by e_1 and ζ and let $[e_1]$ denote the slice in B_N through 0 and e_1 . If $\eta \neq e_1$, then φ maps the slice $L(e_1, \zeta)$ onto $L(e_1, \eta)$ with a boundary fixed point e_1 . When ζ is on the slice $[e_1]$, since $\varphi(0) = 0$, we see that $L(e_1, \zeta) = L(e_1, \eta) = [e_1]$. Thus, φ restricting to $[e_1]$ must be identity, which contradicts to $p = 0$. Otherwise, assume that τ_1 and τ_2 are automorphisms of B_N fixing e_1 , and they map $[e_1]$ onto $L(e_1, \zeta)$ and $L(e_1, \eta)$ onto $[e_1]$ respectively, such that the restriction of $\tau_2 \circ \varphi \circ \tau_1$ to the slice $[e_1]$ is a linear fractional self-map of this disk onto itself with only one boundary fixed point e_1 . Moreover, its boundary boundary dilation coefficient at e_1 satisfies $\alpha \geq d_\varphi(e_1)$. Note that φ is an elliptic linear fractional map of B_N with $\varphi(0) = 0$, which gives $d_\varphi(e_1) > 1$. Thus, we have $\alpha > 1$. It is impossible for an automorphism of the disk with only one boundary fixed point.

If $\eta = e_1$, we see that $\varphi(e_1) = \varphi(\zeta) = e_1$ and φ maps $L(e_1, \zeta)$ into a slice S in B_N through e_1 . Then the restriction of $\tau_2 \circ \varphi \circ \tau_1$ to $[e_1]$ is a linear fractional self-map of this disk, where τ_1 and τ_2 are automorphisms of B_N mapping $[e_1]$ onto $L(e_1, \zeta)$ and S onto $[e_1]$ with $\tau_1(e_1) = \tau_2(e_1) = e_1$. Suppose $\tau_1(\alpha) = \zeta$, then $\tau_2 \circ \varphi \circ \tau_1|_{[e_1]}$ is a

linear fractional map of the disk mapping two boundary points e_1 and α to e_1 . So it is a constant with modulus 1. Therefore, $|\tau_2 \circ \varphi \circ \tau_1(z)| = 1$ for any point $z \in [e_1]$, that is, $|\varphi(z)| = 1$ for any $z \in L(e_1, \zeta)$, which contradicts to the fact $\varphi(B_N) \subset B_N$. Thus, we show the claim.

Next, we will estimate the essential norm of C_φ . Since $\|C_\varphi\|_e = \|C_\varphi^*\|_e = \inf\{\|C_\varphi^* - F\| : F \text{ is compact}\}$, it is easy to see that

$$\begin{aligned} \|C_\varphi\|_e &\geq \limsup_{|z| \rightarrow 1} \left\| C_\varphi^* \frac{K_z}{\|K_z\|} \right\| = \limsup_{|z| \rightarrow 1} \frac{\|K_{\varphi(z)}\|}{\|K_z\|} \\ &= \limsup_{|z| \rightarrow 1} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{N/2}, \end{aligned}$$

where $\|T\|_e$ denotes the essential norm of an operator T and $K_z(w) = \frac{1}{(1 - \langle z, w \rangle)^N}$ is the reproducing kernel of $H^2(B_N)$.

We introduce the pullback measure τ_φ on $\overline{B_N}$ defined by $\tau_\varphi(E) = \sigma[(\varphi^*)^{-1}(E)]$ for Borel subsets E of $\overline{B_N}$, where φ^* denotes the radial limit of φ . Then

$$\int_{\partial B_N} f \circ \varphi^*(\zeta) d\sigma(\zeta) = \int_{B_N} f d\tau_\varphi$$

for every Borel function $f \geq 0$ on $\overline{B_N}$. Let I_{τ_φ} denote the densely defined inclusion operator of $H^2(B_N)$ into $L^2(\tau_\varphi)$. Using Theorem 5.1 of [7], we get

$$\|C_\varphi\|_e^2 = \|I_{\tau_\varphi}\|_e^2 \cong \|\tau_\varphi\|_{e,C},$$

where the ‘‘essential Carleson norm’’ is defined by

$$\|\tau_\varphi\|_{e,C} = \limsup_{t \rightarrow 0} \sup_{\zeta \in \partial B_N} \frac{\tau_\varphi(\Omega(\zeta, t))}{\sigma(Q(\zeta, t))}$$

with $\Omega(\zeta, t) = \{z \in \overline{B_N} : |1 - \langle z, \zeta \rangle| < t\}$ and $Q(\zeta, t) = \Omega(\zeta, t) \cap \partial B_N$.

Since $\varphi(e_1) = e_1$ and φ maps any boundary point $\zeta \neq e_1$ into B_N , this gives

$$\|\tau_\varphi\|_{e,C} = \limsup_{t \rightarrow 0} \frac{\tau_\varphi(\Omega(e_1, t))}{\sigma(Q(e_1, t))} = \limsup_{t \rightarrow 0} \frac{\sigma[(\varphi^*)^{-1}(\Omega(e_1, t))]}{\sigma(Q(e_1, t))}.$$

Applying similar arguments as for proving $\varphi(Q(\zeta, t)) \subset \Omega(\eta, At)$ with $\varphi(\zeta) = \eta$ in Lemma 3.40 of [11], where A is a constant depending on $d_\varphi(\zeta)$, we can calculate that

$$\|\tau_\varphi\|_{e,C} = \limsup_{t \rightarrow 0} \frac{\sigma[(\varphi^*)^{-1}(\Omega(e_1, t))]}{\sigma(Q(e_1, t))} = \limsup_{z \rightarrow e_1} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^N$$

because of $\sigma(Q(e_1, t)) \sim t^N$. Therefore,

$$\|C_\varphi\|_e^2 \leq C \|\tau_\varphi\|_{e,C} \leq C \limsup_{|z| \rightarrow 1} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^N$$

holds for a positive constant C . Note that the essential spectral radius of C_φ can be computed by $r_e(C_\varphi) = \lim_{n \rightarrow \infty} \|C_\varphi^n\|_e^{1/n} = \lim_{n \rightarrow \infty} \|C_{\varphi^n}\|_e^{1/n}$ and we have obtained that

$$\|C_{\varphi^n}\|_e \cong \limsup_{|z| \rightarrow 1} \left(\frac{1 - |z|^2}{1 - |\varphi^n(z)|^2} \right)^{N/2}.$$

Thus, we get

$$r_e(C_\varphi) = \lim_{n \rightarrow \infty} \|C_{\varphi^n}\|_e^{1/n} = \lim_{n \rightarrow \infty} \left(\limsup_{|z| \rightarrow 1} \left(\frac{1 - |z|^2}{1 - |\varphi^n(z)|^2} \right)^{N/2} \right)^{\frac{1}{n}}.$$

□

Kamowitz [17] was the first to investigate the spectrum of C_φ on $H^2(D)$ when φ has an interior fixed point a , he proved that $\sigma(C_\varphi) = \{\lambda : |\lambda| \leq r_e(C_\varphi)\} \cup \{\varphi'(a)^n : n = 1, 2, \dots\} \cup \{1\}$ if φ is analytic in a neighborhood of D , not an inner function. In [10] Cowen and MacCluer obtained the same spectral structure for C_φ when φ is univalent, not automorphism, and $\varphi(a) = a \in B_N$. From their results one can easily deduce the spectrum of C_φ if φ is a linear fractional map of D with an interior and a boundary fixed points (see Theorem A). Moreover, the method of Cowen and MacCluer was used by many authors to study the spectrum of C_φ on other spaces for φ in the same case, For example, on $H^\infty(D)$, the Bloch space and BMOA in one variable, see [32], [26], [24]; for a generalization on $H^\infty(B_E)$ and $H^\infty(B_N)$ see [13] and [31], where B_E is an open unit ball on a complex Banach space. We will use same ideas and approaches suggested by the work of Kamowitz and that of Cowen and MacCluer to obtain the following result.

Theorem 2.5. *Let φ be the same as in Lemma 2.4. Then the spectrum of C_φ on $H^2(B_N)$ is*

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \rho\} \cup \{\text{all possible products of the eigenvalues of } \varphi'(z_0)\} \cup \{1\},$$

where ρ is the essential spectral radius of C_φ .

As in the proof of Lemma 2.4, we may assume $\varphi(0) = 0$ and $\varphi(e_1) = e_1$. For non-negative integers m , let \mathcal{H}_m be the subspace of $H^2(B_N)$ spanned by the monomials of total degree greater than or equal to m , that is, any function $f \in \mathcal{H}_m$ can be written as $f = \sum_{|s| \geq m} f_s$, where f_s is a homogeneous polynomial of degree s . Obviously, \mathcal{H}_m is invariant for C_φ . A easy computation shows that the reproducing kernel in \mathcal{H}_m at the point w is

$$K_w^m(z) = \sum_{|s| \geq m} \frac{(N-1+s)!}{(N-1)!s!} \langle z, w \rangle^s.$$

In particular, we have

$$\|K_w^m\|^2 = \sum_{|s| \geq m} \frac{(N-1+s)!}{(N-1)!s!} |w|^{2s}$$

and

$$C'_{N,m}|w|^m\|K_w\| \leq \|K_w^m\| \leq C''_{N,m}|w|^m\|K_w\|,$$

where $C'_{N,m}$ and $C''_{N,m}$ are positive constants only depending on N and m .

Just as in the disk, our generalization for Proposition 7.32 of [11] to the unit ball is the following lemma. We only need a similar argument used in computing the spectrum of a compact composition operator on $H^2(B_N)$, so the proof will be omitted.

Lemma 2.6. *For φ as in Lemma 2.4, the spectrum of C_φ on $H^2(B_N)$ contains 1 and all possible products of the eigenvalues of $\varphi'(z_0)$. Moreover, if $\lambda \neq 0$ is an eigenvalue of C_φ , then λ will be a product of the eigenvalues of $\varphi'(z_0)$.*

We say the sequence of points $\{z_k\}_{-K}^\infty$ is an iteration sequence for φ if $\varphi(z_k) = z_{k+1}$ for $k \geq -K$. In the proof of Theorem 2.5, the following fact will be needed.

Lemma D. [10] *If φ maps the unit ball into itself, $\varphi(0) = 0$ and φ is not unitary on any slice in B_N . Suppose $r < 1$ is given and $\{z_k\}_{-K}^\infty$ is any iteration sequence with $|z_n| \leq r$. Then there exists $c < 1$ such that $\frac{|z_{k+1}|}{|z_k|} \leq c$ for all $k \geq n$.*

Lemma E. [10] *For φ in Lemma D and $0 < r < 1$. There exists $M < \infty$ such that if $\{z_k\}_{-K}^\infty$ is an iteration sequence with $|z_l| \geq r$ for some $l \geq 0$ and if $\{w_k\}_{-K}^\infty$ is arbitrary, then there is h in $H^\infty(B_N)$ such that $h(z_k) = w_k$ for $-K \leq k \leq l$ and $\|h\|_\infty \leq M \sup\{|w_k| : -K \leq k \leq l\}$.*

Proof of Theorem 2.5. Using Lemma 2.6 and the fact that the spectrum is closed, it suffices to prove

$$\{\lambda : |\lambda| \leq \rho\} \subset \sigma(C_\varphi).$$

Let $C_m = C_{\varphi|_{\mathcal{H}_m}}$ and assume $0 < |\lambda| < \rho$. By Lemma 7.17 of [11], if we can show that λ is in the spectrum of C_m for some positive integer m , then we get the desired conclusion. So we will try to find a positive integer m such that $C_m^* - \lambda I$ is not bounded from below.

Fixing δ with $0 < \delta < 1$, suppose we have an iteration sequence $\{z_k\}_{-K}^\infty$ with $|z_0| > \delta$. Let $n = \max\{k : |z_k| \geq \delta\}$. By Lemma D, we can choose c with $\sqrt{\delta} < c < 1$ such that $|z_{k+1}| < c|z_k|$ for $|z_k| < \sqrt{\delta}$. On the other hand, if $|z_n| > \sqrt{\delta} > \delta$, then $|z_{n+1}| < \delta < \sqrt{\delta}|z_n| < c|z_n|$. Thus, we have $|z_{k+1}| < c|z_k|$ for all $k \geq n$. By induction, $|z_k| < c^{k-n}|z_n|$ holds for all $k \geq n$.

Now, let's define $L_\lambda = \sum_{k=0}^\infty \lambda^{-(k+1)} K_{z_k}^m$, we will see that L_λ is well defined and it is bounded. For $k > n$, $|z_k| < \delta$ gives that

$$\|K_{z_k}^m\| \leq C''_{N,m}|z_k|^m\|K_{z_k}\| = C''_{N,m} \frac{|z_k|^m}{(1 - |z_k|)^{N/2}} \leq C|z_k|^m,$$

where C is a positive constant depending on N, m and δ . It follows that

$$\sum_{k=n+1}^{\infty} |\lambda|^{-(k+1)} \|K_{z_k}^m\| \leq C \sum_{k=n+1}^{\infty} \frac{|z_k|^m}{|\lambda|^{k+1}} \leq C \frac{|z_n|^m}{|\lambda|^{n+1}} \sum_{k=n+1}^{\infty} \left(\frac{c^m}{|\lambda|} \right)^{k-n}.$$

Since $c < 1$, if we choose m so large that $c^m < |\lambda|$, then the series defining L_λ converges.

Next, we will estimate $\|(C_m^* - \lambda I)L_\lambda\|/\|L_\lambda\|$. Since \mathcal{H}_m is invariant for C_φ , it is easy to see $C_\varphi^* K_w^m = K_{\varphi(w)}^m$ and so

$$\begin{aligned} (C_m^* - \lambda I)L_\lambda &= (C_m^* - \lambda I) \sum_{k=0}^{\infty} \lambda^{-(k+1)} K_{z_k}^m = \sum_{k=0}^{\infty} \left(\lambda^{-(k+1)} K_{\varphi(z_k)}^m - \lambda^{-k} K_{z_k}^m \right) \\ &= \sum_{k=0}^{\infty} \left(\lambda^{-(k+1)} K_{z_{k+1}}^m - \lambda^{-k} K_{z_k}^m \right) = -K_{z_0}^m. \end{aligned}$$

This means $\|(C_m^* - \lambda I)L_\lambda\| = \|K_{z_0}^m\|$. It remains to give a lower bound for $\|L_\lambda\|$.

Choose an m -homogenous polynomial $P(z)$ on B_N satisfying $\|P\|_\infty = 1$ and $|P(z_n)| = |z_n|^m$. It is clear that $|P(z)/|z|^m| = |P(z/|z|)| \leq 1$ for any $z \in B_N$, which gives $|P(z)| \leq |z|^m$. Applying Lemma E to the iteration sequence $\{z_k\}_{-K}^\infty$, there is a function $g \in H^\infty(B_N)$ such that $\|g\|_\infty \leq M$, $g(z_k) = 0$ for $0 \leq k < n$, and $g(z_n) = 1$. For $f = \frac{Pg}{(1 - \langle z, z_n \rangle)^N}$, we obtain

$$\langle L_\lambda, f \rangle = \left\langle \sum_{k=0}^{\infty} \lambda^{-(k+1)} K_{z_k}^m, f \right\rangle = \lambda^{-(n+1)} \overline{f(z_n)} + \sum_{k=n+1}^{\infty} \lambda^{-(k+1)} \overline{f(z_k)}.$$

Note that $|z_k| < \delta$ for $k > n$, so $|1 - \langle z_k, z_n \rangle|^N \geq (1 - |z_k|)^N \geq C(\delta)$. Choose m large enough so that $c^m/|\lambda| < \frac{1}{2^{N+2}M}$, we calculate that

$$\begin{aligned} \left| \sum_{k=n+1}^{\infty} \lambda^{-(k+1)} \overline{f(z_k)} \right| &= \left| \sum_{k=n+1}^{\infty} \frac{1}{\lambda^{k+1}} \frac{\overline{P(z_k)g(z_k)}}{(1 - \langle z_k, z_n \rangle)^N} \right| \leq \sum_{k=n+1}^{\infty} \frac{|P(z_k)g(z_k)|}{|\lambda|^{k+1} |1 - \langle z_k, z_n \rangle|^N} \\ &\leq \sum_{k=n+1}^{\infty} \frac{M|z_k|^m}{|\lambda|^{k+1} |1 - \langle z_k, z_n \rangle|^N} \leq \frac{M|z_n|^m}{C(\delta)|\lambda|^{n+1}} \sum_{k=n+1}^{\infty} \left(\frac{c^m}{|\lambda|} \right)^{k-n} \\ &\leq \frac{|z_n|^m}{2^{N+1}C(\delta)|\lambda|^{n+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle L_\lambda, f \rangle| &\geq \left| \lambda^{-(n+1)} \overline{f(z_n)} \right| - \left| \sum_{k=n+1}^{\infty} \lambda^{-(k+1)} \overline{f(z_k)} \right| \\ &\geq \frac{1}{2} \frac{|P(z_n)g(z_n)|}{|\lambda|^{n+1} (1 - |z_n|^2)^N} + \frac{1}{2} \frac{|P(z_n)g(z_n)|}{|\lambda|^{n+1} (1 - |z_n|^2)^N} - \frac{|z_n|^m}{2^{N+1}C(\delta)|\lambda|^{n+1}} \\ &= \frac{|z_n|^m}{2|\lambda|^{n+1} (1 - |z_n|^2)^N} + \frac{|z_n|^m}{2|\lambda|^{n+1} (1 - |z_n|^2)^N} - \frac{|z_n|^m}{2^{N+1}C(\delta)|\lambda|^{n+1}} \\ &\geq \frac{|z_n|^m}{2|\lambda|^{n+1} (1 - |z_n|^2)^N}, \end{aligned}$$

where we have used the fact that $|z_n| \geq \delta$ gives $(1 - |z_n|^2)^N \leq 2^N C(\delta)$ to obtain the third inequality. Since

$$\|f\| = \left\| \frac{Pg}{(1 - \langle z, z_n \rangle)^N} \right\| \leq \frac{M}{(1 - |z_n|^2)^{N/2}},$$

we deduce that

$$\begin{aligned} \|L_\lambda\| &\geq \frac{|\langle L_\lambda, f \rangle|}{\|f\|} \geq \frac{\frac{|z_n|^m}{2|\lambda|^{n+1}} \frac{1}{(1 - |z_n|^2)^N}}{\frac{M}{(1 - |z_n|^2)^{N/2}}} \\ &= \frac{|z_n|^m}{2M|\lambda|^{n+1}} \frac{1}{(1 - |z_n|^2)^{N/2}} = \frac{|z_n|^m \|K_{z_n}\|}{2M|\lambda|^{n+1}} \\ &\geq \frac{\|K_{z_n}^m\|}{2MC'_{N,m}|\lambda|^{n+1}}. \end{aligned}$$

Hence,

$$\frac{\|(C_m^* - \lambda I)L_\lambda\|}{\|L_\lambda\|} \leq \frac{2MC'_{N,m}|\lambda|^{n+1}\|K_{z_0}^m\|}{\|K_{z_n}^m\|}.$$

When m is fixed, we see that $\|K_w^m\| \leq \|K_w\| \leq \|K_w^m\| + C_m$ for all $w \in B_N$, which gives

$$\frac{\|K_{\varphi^n(w)}^m\|}{\|K_w^m\|} \approx \frac{\|K_{\varphi^n(w)}\|}{\|K_w\|}.$$

On the other hand, by Lemma 2.4, the essential spectral radius of C_φ is

$$\rho = \lim_{n \rightarrow \infty} \left(\limsup_{|w| \rightarrow 1} \left(\frac{1 - |w|^2}{1 - |\varphi^n(w)|^2} \right)^{N/2} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\limsup_{|w| \rightarrow 1} \frac{\|K_{\varphi^n(w)}\|}{\|K_w\|} \right)^{\frac{1}{n}}.$$

The above arguments show that

$$\rho = \lim_{n \rightarrow \infty} \left(\limsup_{|w| \rightarrow 1} \frac{\|K_{\varphi^n(w)}^m\|}{\|K_w^m\|} \right)^{\frac{1}{n}}.$$

Since $\rho > 0$, we have $\limsup_{|w| \rightarrow 1} \|K_{\varphi^n(w)}^m\| = \infty$ for every n . Thus, for any positive integer n , there exist points w near ∂B_N such that $|\varphi^n(w)| > \delta$. Moreover, there exists ρ' with $|\lambda| < \rho' < \rho$ such that

$$\frac{\|K_{\varphi^n(w)}^m\|}{\|K_w^m\|} \geq (\rho')^n.$$

Let $z_0 = w$ and $z_{k+1} = \varphi(z_k)$ for $k \geq 0$. It is clear that the iteration sequence $\{z_k\}_0^\infty$ satisfies $|z_0| > |z_n| > \delta$. Consequently, for this iteration sequence, we obtain

$$\begin{aligned} \frac{\|(C_m^* - \lambda I)L_\lambda\|}{\|L_\lambda\|} &= \frac{\|(C_m^* - \lambda I) \sum_{k=0}^\infty \lambda^{-(k+1)} K_{z_k}^m\|}{\|\sum_{k=0}^\infty \lambda^{-(k+1)} K_{z_k}^m\|} \\ &\leq \frac{2MC'_{N,m}|\lambda|^{n+1}\|K_{z_0}^m\|}{\|K_{z_n}^m\|} \leq 2MC'_{N,m}|\lambda| \left(\frac{|\lambda|}{\rho'} \right)^n. \end{aligned}$$

Note that $|\lambda| < \rho'$, we may form iteration sequences for which n is sufficiently large, so that the above inequality is sufficiently small. This forces that $C_m^* - \lambda I$ is not bounded below. Hence, $C_m^* - \lambda I$ is not invertible and so we complete the proof. \square

3 Spectra of hyperbolic composition operators

Let φ be a hyperbolic linear fractional self-map of B_N , it may have one or two boundary fixed points. This implies that the associated composition operator C_φ may have different characterization for its spectrum on $H^2(B_N)$. First, we will find a conjugated form of φ on the Siegel half-plane for each case.

Recall that the unit ball B_N is biholomorphic to the Siegel half-plane $H_N = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : \operatorname{Re} z > |w|^2\}$ via the Cayley transform σ_C defined by

$$\sigma_C(z, w) = \left(\frac{1+z}{1-z}, \frac{w}{1-z} \right), \quad (z, w) \in \mathbb{C} \times \mathbb{C}^{N-1}.$$

It extends to a homeomorphism of $\overline{B_N}$ onto $H_N \cup \partial H_N \cup \{\infty\}$, the one-point compactification of $\overline{H_N}$. Its reciprocal is given by

$$\sigma_C^{-1}(z, w) = \left(\frac{z-1}{z+1}, \frac{2w}{z+1} \right).$$

Let φ be a linear fractional map of B_N with Denjoy-Wolff point e_1 and boundary dilatation coefficient α . Applying Proposition 4.2 of [6], φ is conjugated to a map ψ on H_N with the form

$$\psi(z, w) = \frac{1}{\alpha}(z + \langle w, b \rangle + c, Aw + d), \quad (z, w) \in H_N,$$

where $c \in \mathbb{C}$, $b, d \in \mathbb{C}^{N-1}$ and $A \in \mathbb{C}^{(N-1) \times (N-1)}$, which satisfy $\alpha \operatorname{Re} c \geq |d|^2$ and $\|A\| \leq \sqrt{\alpha}$. If φ is hyperbolic, then $\alpha < 1$. Combining this with $\|A^*\| = \|A\| \leq \sqrt{\alpha} < 1$, we see that $A^* - I$ is invertible. Thus, there exists a vector $k_1 \in \mathbb{C}^{N-1}$ such that $b = 2(A^* - I)k_1$. Consider the following Heisenberg transformation

$$\eta(z, w) = (z + 2\langle w, k_1 \rangle + k_2, w + k_1), \quad (z, w) \in H_N,$$

with $\operatorname{Re} k_2 = |k_1|^2$, it is an automorphism of H_N . Conjugating ψ by η , we obtain

$$\phi(z, w) = \eta^{-1} \circ \psi \circ \eta(z, w) = \frac{1}{\alpha}(z + c', Aw + d'), \quad (z, w) \in H_N,$$

where $c' \in \mathbb{C}$, $d' \in \mathbb{C}^{N-1}$ and $\alpha \operatorname{Re} c' \geq |d'|^2$. If $c' = c_1 + ic_2$ with $c_1, c_2 \in \mathbb{R}$, let $\nu(z, w) = (z - \frac{ic_2}{1-\alpha}, w)$, then ν is an automorphism of H_N and

$$\nu^{-1} \circ \phi \circ \nu(z, w) = \frac{1}{\alpha}(z + c_1, Aw + d'), \quad (z, w) \in H_N,$$

with $\alpha c_1 \geq |d'|^2$. Thus, we may assume $c' \in \mathbb{R}$.

Therefore, similar to the classification of semi-group of hyperbolic linear fractional self-maps of B_N in [6], we have the following result.

Proposition 3.1. *Let φ is a hyperbolic linear fractional self-map of B_N with the boundary dilatation coefficient α .*

(i) *If φ fixes only one boundary point, then φ is conjugated to a map ϕ of the form*

$$\phi(z, w) = \frac{1}{\alpha}(z + c, Aw + d), \quad (z, w) \in H_N,$$

where $c \in \mathbb{R}$, $d \in \mathbb{C}^{N-1}$ with $\alpha c \geq |d|^2$ and $A \in \mathbb{C}^{(N-1) \times (N-1)}$ satisfying $\|A\| \leq \sqrt{\alpha}$.

(ii) *If φ fixes two boundary points, then φ is conjugated to a map ϕ of the form*

$$\phi(z, w) = \left(\frac{z}{\alpha}, \frac{Aw}{\sqrt{\alpha}}\right), \quad (z, w) \in H_N,$$

where A is a matrix of order $N - 1$ with $\|A\| \leq 1$.

Proof. The previous arguments give that φ is conjugated to a map on the Siegel half-plane H_N with the form

$$\phi(z, w) = \frac{1}{\alpha}(z + c, Aw + d),$$

where $c \in \mathbb{R}$ with $\alpha c \geq |d|^2$ and $\|A\| \leq \sqrt{\alpha}$. If φ fixes only one boundary point, then $c \neq 0$. To see this, we may assume $c = 0$. It is clear that $d = 0$ from $|d|^2 \leq \alpha c$. A calculation shows that the conjugate map ϕ of φ fixes two boundary points 0 and ∞ . So φ must have two boundary fixed points and we get a contradiction.

If φ has two boundary fixed points, so is its conjugate map ϕ . Let $(z, w) \in \partial H_N$ be another fixed point of ϕ different from its Denjoy-Wolff point. We see that $\operatorname{Re} z = |w|^2$, $z + c = \alpha z$ and $Aw + d = \alpha w$. Since $c \in \mathbb{R}$ and $\alpha < 1$, then $c = (\alpha - 1)z$ gives $c \leq 0$. On the other hand, we have $\alpha c \geq |d|^2 \geq 0$. Thus, $c = d = 0$ and so the conjugate map ϕ of φ has the form

$$\phi(z, w) = \frac{1}{\alpha}(z, Aw) = \left(\frac{z}{\alpha}, \frac{A'w}{\sqrt{\alpha}}\right)$$

with $\|A'\| = \|\frac{1}{\sqrt{\alpha}}A\| \leq 1$. \square

Now, we will compute the spectra of hyperbolic composition operators on $H^2(B_N)$. For the case φ fixing only one boundary point, we need the technique used in determining the spectra of composition operators on some weighted Hardy spaces, whose symbols are hyperbolic automorphisms of B_N (see Theorem 7.7 of [11]). Moreover, the spectrum of C_φ has similar properties as that in the disk. This result has been obtained on some weighted Hardy spaces in the first author's doctoral thesis [19].

Theorem 3.2. *Let $\varphi \in \operatorname{LFM}(B_N)$ be hyperbolic, non-automorphism. If φ has only one boundary fixed point with the boundary dilatation coefficient α . Then the spectrum and the essential spectrum of C_φ on $H^2(B_N)$ are the closed disk $|\lambda| \leq \alpha^{-N/2}$.*

Moreover, every point in the disk $|\lambda| < \alpha^{-N/2}$ is an eigenvalue of C_φ of infinite multiplicity.

Proof. Proposition 3.1 tells us that φ is conjugated to a map

$$\phi(z, w) = \frac{1}{\alpha}(z + c, Aw + d), \quad (z, w) \in H_N,$$

with $\alpha c \geq |d|^2$ and $\|A\| \leq \sqrt{\alpha}$. Transferring back to the unit ball by the Cayley transform σ_C , we have

$$\begin{aligned} \psi(z, w) &= \sigma_C^{-1} \circ \phi \circ \sigma_C(z, w) \\ &= \left(\frac{1 + c - \lambda + (1 - c + \lambda)z}{1 + c + \lambda + (1 - c - \lambda)z}, \frac{2Aw + 2(1 - z)d}{1 + c + \lambda + (1 - c - \lambda)z} \right), \quad (z, w) \in B_N. \end{aligned}$$

Next, we will determine the spectrum of C_ψ . We first prove that each point in the disk $|\lambda| < \alpha^{-N/2}$ is an eigenvalue of C_ψ of infinite multiplicity. For $z \in D$, define

$$\widetilde{\psi}_1(z) = \frac{1 + c - \lambda + (1 - c + \lambda)z}{1 + c + \lambda + (1 - c - \lambda)z}$$

and set

$$\widetilde{\beta}(k)^2 = \frac{(N - 1)!k!}{(N - 1 + k)!}, \quad k = 1, 2, \dots$$

If we set $\widehat{\beta}(k)^2 = (N - 1)!(k + 1)^{1-N}$, then $\widehat{\beta}(k)^2 \geq \widetilde{\beta}(k)^2$ and $H^2(\widehat{\beta}, D) \subset H^2(\widetilde{\beta}, D)$, where $H^2(\widehat{\beta}, D)$ and $H^2(\widetilde{\beta}, D)$ respectively denote the weighted Hardy spaces with the weights $\widehat{\beta}$ and $\widetilde{\beta}$, see Section 4 for their definitions.

Since $c > 0$, the linear fractional self-map $\widetilde{\psi}_1$ of D fixes one boundary point 1 and the other point $z_0 = \frac{c+1-\lambda}{c-1+\lambda}$ with $|z_0| > 1$. In Theorem 8 of [16], Hurst has investigated the spectrum of $C_{\widetilde{\psi}_1}$ on the weighted Hardy space $H^2(\beta, D)$ with the weight $\beta(k) = (k + 1)^\alpha$ ($\alpha \leq 0$), and proved that each point of the disk $|\lambda| < (\widetilde{\psi}_1)'(1)^{(2\alpha-1)/2}$ is an eigenvalue of $C_{\widetilde{\psi}_1}$ of infinite multiplicity. It is clear that $1 - N \leq 0$. Therefore, using Hurst's result, if λ is in the disk $|\lambda| < (\widetilde{\psi}_1)'(1)^{-N/2}$, we may find infinitely many linearly independent functions f in $H^2(\widehat{\beta}, D) \subset H^2(\widetilde{\beta}, D)$ such that $f \circ \widetilde{\psi}_1 = \lambda f$. For $(z, w) \in B_N$, define the extension operator by $Ef(z, w) = f(z)$, by Proposition 2.21 of [11], E is an isometry of $H^2(\widetilde{\beta}, D)$ into $H^2(B_N)$. Note that $\widetilde{\psi}_1(z) = \psi_1(z, w)$ and $(\widetilde{\psi}_1)'(1) = \alpha$, where ψ_1 is the first coordinate of ψ . Thus, we have $Ef \in H^2(B_N)$ and

$$\begin{aligned} Ef \circ \psi(z, w) &= Ef(\psi_1(z, w), \dots, \psi_N(z, w)) = f \circ \psi_1(z, w) \\ &= f \circ \widetilde{\psi}_1(z) = \lambda f(z) = \lambda Ef(z, w). \end{aligned}$$

This shows that λ is an eigenvalue of C_ψ on $H^2(B_N)$ of infinite multiplicity.

By Theorem C, we see that the spectral radius of C_ψ is $\alpha^{-N/2}$. Hence, the spectrum of C_ψ is contained in the disk $|\lambda| \leq \alpha^{-N/2}$. Since the essential spectrum

is contained in the spectrum and contains all eigenvalues of infinite multiplicity, we obtain the desired conclusion. \square

Finally, when a hyperbolic linear fractional map φ fixes two boundary points, the first author [19] has proved that each point of the annulus $\alpha^{N/2} < |\lambda| < \alpha^{-N/2}$ is an eigenvalue of C_φ on $H^2(B_N)$ of infinite multiplicity, and its spectrum is contained in the disk $|\lambda| \leq \alpha^{-N/2}$. Applying a different approach, Jury [23] obtained the same result. Recently, the spectrum of C_φ has been determined by Xu and Deng [30], the main idea also comes from Bayart [4]. By Theorem 3.1, we see that φ is conjugated to a map on H_N with the form $\phi(z, w) = (\frac{z}{\alpha}, \frac{Aw}{\sqrt{\alpha}})$, then the spectrum of C_φ is as follows.

Theorem F. *Let $\varphi \in \text{LFM}(B_N)$ be hyperbolic, non-automorphism. If φ fixes two boundary points with the boundary dilation coefficient α , then the spectrum of C_φ on $H^2(B_N)$ is*

$$\sigma(C_\varphi) = \bigcup_{\beta \in \mathbb{N}^{N-1}} \Lambda^\beta S \cup \{0\},$$

where Λ^β denotes $\lambda_1^{\beta_1} \cdots \lambda_{N-1}^{\beta_{N-1}}$ with $\lambda_1, \dots, \lambda_{N-1}$ the eigenvalues of A and S is the annulus $\{\lambda : \alpha^{N/2} < |\lambda| < \alpha^{-N/2}\}$.

4 Concluding remarks

Now, the spectra of all linear fractional composition operators on $H^2(B_N)$ have been completely determined. Moreover, some works are adapted to other spaces, for example, the spectra of parabolic composition operators have been generalized to weighted Bergman spaces by Bayart [4].

Let's recall that the weighted Bergman space $A_\alpha^2(B_N)$ is the space of all analytic functions f in B_N such that

$$\|f\|_{A_\alpha^2}^2 = \int_{B_N} |f(z)|^2 (1 - |z|^2)^\alpha dv(z) < \infty$$

for $\alpha > -1$. For the notational conveniences, we set $A_{-1}^2(B_N) = H^2(B_N)$. Let $f(z) = \sum_{k=0}^\infty f_k(z)$ be the homogeneous expansion of f . For $s \geq 0$, the fractional derivative of f order s is defined by

$$R^s f(z) = \sum_{k=1}^\infty k^s f_k(z).$$

Thus, for $\alpha \geq -1$ and $s \geq 0$, the holomorphic Sobolev space $A_{\alpha,s}^2$ is defined as

$$A_{\alpha,s}^2(B_N) = \{f \in H(B_N) : R^s f \in A_\alpha^2(B_N)\}$$

with the norm $\|f\|_{A_{\alpha,s}^2} = \|R^s f\|_{A_\alpha^2} + |f(0)|$. Koo and Park [18] proved that C_φ is bounded on $A_{\alpha,s}^2(B_N)$ if and only if φ satisfies Wogen's condition for $\varphi \in C^{s+4}(\overline{B_N})$. As we know, any linear fractional self-map of B_N satisfies Wogen's condition (see

[12]). This implies that all linear fractional composition operators are bounded on $A_{\alpha,s}^2(B_N)$.

Let $H^2(\beta, B_N)$ be the weighted Hardy space with the weight $\beta(k) = (k+1)^\nu$, where ν is a real number. Its norm is given by

$$\|f\|_{H^2(\beta)}^2 = \sum_{k=0}^{\infty} \|f_k\|^2 \beta(k)^2.$$

It will be convenient to work with an equivalent norm on $H^2(\beta, B_N)$, the following lemma is a generalization of Lemma 1.2 in [14] to the unit ball.

Lemma 4.1. *Suppose that ν is a real number and $s \geq 0$ is an integer such that $s \geq \nu$. Then the norm of the weighted Hardy space $H^2(\beta, B_N)$ with the weight $\beta(k) = (k+1)^\nu$ is equivalent to the norm of the Sobolev space $A_{2s-2\nu-1,s}^2$.*

Proof. Let $f = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha z^\alpha$ be in $H^2(\beta, B_N)$. We compute that

$$\begin{aligned} \|R^s f\|_{A_c^2}^2 &= \int_{B_N} \left| \sum_{k=1}^{\infty} \sum_{|\alpha|=k} k^s a_\alpha z^\alpha \right|^2 (1-|z|^2)^c dv(z) \\ &= \sum_{k=1}^{\infty} \sum_{|\alpha|=k} k^{2s} |a_\alpha|^2 \int_0^1 r^{2k} (1-r^2)^c r dr \int_{\partial B_N} |\zeta^\alpha|^2 d\sigma(\zeta) \\ &= \sum_{k=1}^{\infty} \sum_{|\alpha|=k} k^{2s} |a_\alpha|^2 \frac{(N-1)!\alpha!}{(N-1+k)!} \cdot \frac{c!k!}{(c+k+1)!}, \end{aligned}$$

which is equivalent to

$$\sum_{k=1}^{\infty} \sum_{|\alpha|=k} |a_\alpha|^2 \frac{(N-1)!\alpha!}{(N-1+k)!} \cdot k^{2s-c-1}$$

according to Stirling's formula. On the other hand, we have

$$\|f\|_{H^2(\beta)}^2 = \sum_{k=0}^{\infty} \|f_k\|^2 (k+1)^{2\nu} = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} |a_\alpha|^2 \frac{(N-1)!\alpha!}{(N-1+k)!} \cdot (k+1)^{2\nu}.$$

Note that $\|f\|_{A_{c,s}^2} = \|R^s f\|_{A_c^2} + |f(0)|$. If $2s-c-1 = 2\nu$ and $s \geq \nu$, i.e. $c = 2s-2\nu-1$ with $c \geq -1$, then the norm of the weighted Hardy space $H^2(\beta, B_N)$ is comparable with that of the Sobolev space $A_{2s-2\nu-1,s}^2$. \square

Thus, using Lemma 4.1, we have the following result for weighted Hardy spaces.

Proposition 4.2. *Let φ be a linear fractional self-map of B_N . Then C_φ is bounded on the weighted Hardy space $H^2(\beta, B_N)$ with the weight $\beta(k) = (k+1)^\nu$, where ν is a real number.*

Therefore, if we can calculate the spectral radii of linear fractional composition operators on $H^2(\beta, B_N)$ with the weight $\beta(k) = (k+1)^\nu$, one subject of this topic shall be to characterize the spectra of these composition operators.

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